

PROBLEM SECTION

Edited by Nenad P. Cakić

We publish research problems in all areas of Mathematics that fall in one of the following categories:

- a) Research problems or conjectures with a solution not known to the proposer. These problems are marked with an asterisk after their number.
- b) Research problems seeking a new, more elegant solution
- c) Inquiries about references and state of the art regarding a particular problem.

Problems should be submitted in a form that is easy to understand to a nonspecialist in a field. If using special terms or notations can not be avoided, they should be defined in a statement of a problem. A problem may be accompanied by a short comment (addressed primarily to specialists) that explains why the solution could be of an interest.

Solutions should be worked out in all reasonable details.

Correspondence regarding Problem Section should be sent to:

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- **Problem 71*** proposed by WALTHER JANOUS, Innsbruck, Austria.

The sequence $\{z(n), n = 1, 2, 3, \dots\}$ of positive real numbers is defined by

$$z(n) := \frac{1}{\sqrt{n}} \sum_{j=1}^{+\infty} e^{n-j} \cdot \left(\frac{j}{n}\right)^n.$$

Prove the validity of $\lim_{n \rightarrow +\infty} z(n) = \sqrt{2\pi}$.

- **Problem 72*** proposed by WALTHER JANOUS, Innsbruck, Austria.

We consider the function $f(x) = (1-x)^x + x^{1-x}$, where $0 \leq x \leq 1$. Prove the following two inequalities:

i) The maximum exponent μ such that $f(x) \leq 1 + 2(x(1-x))^\mu$ is valid for all $0 \leq x \leq 1$ equals $\mu = \frac{\ln(2+2\sqrt{2})}{2\ln 2}$.

ii) The minimum factor μ such that $f(x) \leq 1 + \mu x(1-x)$ is valid for all $0 \leq x \leq 1$ equals $\mu = 4\sqrt{2} - 4$.

• **Problem 73** proposed by OVIDIU FURDIU, Kalamazoo, USA.

Prove that the following holds:

$$\sum_{n=1}^{+\infty} \frac{\left(1 + \frac{1}{2} + \dots + \frac{1}{n}\right) \left(\zeta(3) - \sum_{k=1}^n \frac{1}{k^3}\right)}{n} = 2\zeta(2)\zeta(3) - \frac{7}{2}\zeta(5),$$

where $\zeta(a)$ is the RIEMANN-Zeta function evaluated at a .

• **Problem 74** proposed by JOSÉ LUIS DÍAZ-BARRERO, Barcelona, Spain.

Let $A(z) = \sum_{k=0}^n a_k z^k$ ($a_k \neq 0$) be nonconstant complex polynomial. Show that all its zeroes lie in the annulus $\mathcal{C} = \{z \in \mathbb{C} : r_1 \leq |z| \leq r_2\}$, where

$$r_1 = \min_{1 \leq k \leq n} \left(\frac{2^k \binom{n}{k}}{3^n - 1} \left| \frac{a_0}{a_k} \right| \right)^{1/k} \quad \text{and} \quad r_2 = \max_{1 \leq k \leq n} \left(\frac{3^n - 1}{2^k \binom{n}{k}} \left| \frac{a_{n-k}}{a_n} \right| \right)^{1/k}.$$

• **Problem 75** proposed by XIAO-MING ZHANG and TIE-QUAN XU, Dalian, China.

If let $x > 0$, then

$$\begin{aligned} 1^\circ \quad & \sum_{n=1}^{+\infty} \frac{nx}{(n+x)^3} < \frac{1}{2}; \quad 2^\circ \quad \sum_{n=1}^{+\infty} \frac{n+1-x}{(n+1+x)^3} < \frac{1}{(x+1)^2}; \\ 3^\circ \quad & \sum_{n=1}^{+\infty} \frac{n+1-x}{(n+1+x)^3} > \frac{x}{(x+1)^3}. \end{aligned}$$

• **Problem 76*** proposed by D. M. MILOŠEVIĆ, Gornji Milanovac, Serbia.

Let a, b, c are the sides, A, B, C the angles (measured in radians), and s the semi-perimeter of a triangle. Prove that

$$\frac{y+z}{x} \cdot \frac{a}{A^\lambda(s-a)} + \frac{z+x}{y} \cdot \frac{b}{B^\lambda(s-b)} + \frac{x+y}{z} \cdot \frac{c}{C^\lambda(s-c)} \geq 12 \cdot \left(\frac{3}{\pi}\right)^\lambda,$$

where x, y, z, λ are positive real numbers and $\lambda \geq 1$.

• **Problem 77.** (Proposed by MIRKO JOVANOVIĆ, Faculty of Electrical Engineering, Belgrade, Serbia.)

Let $a = BC$, $b = CA$, $c = AB$, $a' = AD$, $b' = BD$, $c' = CD$ be the edge lengths of a tetrahedron $ABCD$ and let F, R, r be the total area, the circumradius sphere and inradius sphere respectively of the tetrahedron $ABCD$, then

$$abc + ab'c' + a'bc' + a'b'c \leq 4F\sqrt{R^2 - r^2}.$$

SOLUTIONS

• **Problem 56*** proposed by SLAVKO SIMIĆ, Mathematical Institute, Belgrade, Serbia.

Prove or disprove that the inequality $b > 1 + \log^\beta b$, holds for each $b > 1$ if and only if $\beta \in [1, 5/2]$.

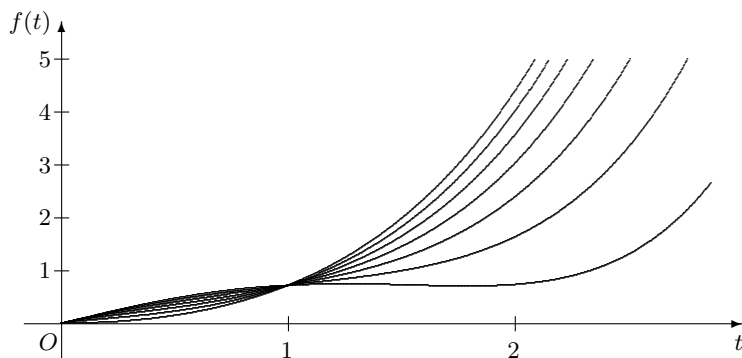
Partial solution by WALTHER JANOUS, Innsbruck, Austria.

Numeric checks are not in favour of the stated conjecture. Indeed, let $b = e^t$, where $t > 0$. Then we get for the inequality under consideration $e^t > t^\beta + 1$. We now put $f(t) := e^t - (1 + t^\beta)$. As in the vicinity of $t = 0$ we have $e^t = 1 + t + \dots$ with $\dots = O(t^2)$, we infer $f(t) = t - t^\beta + \dots$, which readily yields $\beta \geq 1$.

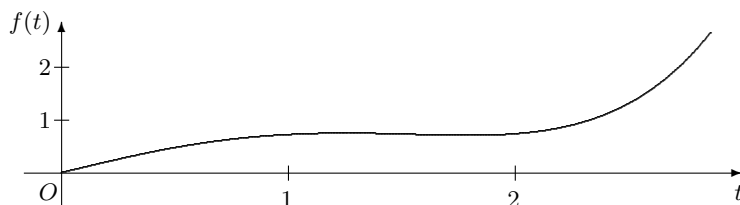
Next, we plot the graphs of $f(t)$ for $\beta = 1$ to $5/2$ step $1/4$. The expressions of these functions are

$$[e^t - t - 1, e^t - t^{5/4} - 1, e^t - t^{3/2} - 1, e^t - t^{7/4} - 1, e^t - t^2 - 1, e^t - t^{9/4} - 1, e^t - t^{5/2} - 1]$$

with their graphs



The graph of $f(t)$ for $\beta = 5/2$, that is



already indicates that the upper bound of the β -interval can be increased.

Now we reproduce the graph of $f(t)$ for $\beta = 13/5$:



The rest is a thorough search and it reveals for the upper bound the value $\beta_{\max} = 2.63274835\dots$

REMARK. To my understanding it is hard (if not even impossible ???) to determine the precise value of the maximum-value suitable β 's.

• **Problem 59** proposed by MIHÁLY BENCZE and MARIAN DINCA, Brasov, Romania.

If $F : \mathbb{R} \rightarrow \mathbb{R}$ is a convex function and $a_k \in \mathbb{R}$ ($k = 1, 2, \dots, n$), $S = \sum_{k=1}^n a_k$, then

$$\sum_{k=1}^n F(S - (n-1)a_k) - \sum_{k=1}^n F(a_k) \geq (n-2) \left(\sum_{k=1}^n F(a_k) - nF\left(\frac{1}{n} \sum_{k=1}^n a_k\right) \right) \geq 0.$$

Solution by MICHEL BATAILLE, Rouen, France.

The result is obvious for $n = 1$ and $n = 2$, so we will suppose $n \geq 3$. The right-hand inequality results directly from JENSEN's inequality applied to the convex function F . The left-hand inequality may be rewritten as

$$(1) \quad \sum_{k=1}^n F(S - (n-1)a_k) + n(n-2)F\left(\frac{1}{n} \sum_{k=1}^n a_k\right) \geq (n-1) \sum_{k=1}^n F(a_k).$$

Let $x_j = S - (n-1)a_j$. Then $x_1 + x_2 + \dots + x_n = S$ and $S - x_j = (n-1)a_j$, $j = 1, 2, \dots, n$, so that (1) becomes

$$(2) \quad \sum_{j=1}^n F(x_j) + n(n-2)F\left(\frac{S}{n}\right) \geq (n-1) \sum_{j=1}^n F\left(\frac{S-x_j}{n-1}\right).$$

The result follows since (2) is just the particular case $k = n - 1$ of POPOVICIU's inequality for the convex function F (see: T. POPOVICIU: *Sur certaines enégalités qui caractérisent les fonctions convexes*. Analele Stiintifice Univ. "Al. I. Cuza" Iasi Sect. I-a Mat. (N.S.) 11 B (1965), 155–164, where T. POPOVICIU proves (2) by approaching F with linear combinations of functions of the form $x \mapsto |x - \rho|$. Inequality (2) is also explicitly cited by MURRAY KLAMKIN in his comment of the solution to problem E3141, Amer. Math. Monthly, Vol 95, No 7, Aug.-Sept. 1988, p. 656).

Also solved by LI ZHOU (Winter Haven, USA) and the proposers. M. BENCZE and M. DINCÁ give the following remark and applications:

REMARK. This is a refinement of inequality $\sum_{k=1}^n F(S - (n-1)a_k) \geq \sum_{k=1}^n F(a_k)$, $F : \mathbb{R} \rightarrow \mathbb{R}$ convex, $a_k \in \mathbb{R}$, $S = \sum_{k=1}^n a_k$ ($k = 1, 2, \dots, n$), obtained by MURRAY S. KLAMKIN in *Extension of an inequality*. Univ. Beograd, Publ. Elektroteh. Fak., Ser. Mat., **7** (1996), 72–73.

APPLICATION 1. If $a_k > 0$, $S = \sum_{k=1}^n a_k$, $S - (n-1)a_k > 0$ ($k = 1, 2, \dots, n$), then

$$\prod_{k=1}^n a_k \geq \prod_{k=1}^n (S - (n-1)a_k) \left(\frac{S^n}{n^n \prod_{k=1}^n a_k} \right)^{n-2} \geq \prod_{k=1}^n (S - (n-1)a_k).$$

In problem we take $F(x) = -\ln x$.

This is a refinement of inequality $\prod_{k=1}^n a_k \geq \prod_{k=1}^n (S - (n-1)a_k)$ obtained by D. S. MITRINOVIĆ and D. D. ADAMOVIĆ in *Analytic Inequalities*. Springer-Verlag, Heidelberg, 1970, pp. 208–209.

APPLICATION 2. If $x_k > 0$ ($k = 1, 2, \dots, n$), then

$$\begin{aligned} \prod_{k=1}^n (x_1 + \dots + x_{k-1} + x_{k+1} + \dots + x_n) &\geq (n-1)^n \sqrt[n-1]{\prod_{k=1}^n x_k} \left(\frac{1}{n} \prod_{k=1}^n x_k \right)^{\frac{n(n-2)}{n-1}} \\ &\geq (n-1)^n \prod_{k=1}^n x_k. \end{aligned}$$

(A refinement of generalized CESARO's inequality).

Proof. In Application 1 we take $a_k = \frac{S - x_k}{n-1}$ ($k = 1, 2, \dots, n$), $S = \sum_{k=1}^n x_k$.

• **Problem 60** proposed by MIHÁLY BENCZE and MARIAN DINCA, Brasov, Romania.

If $f : A \rightarrow (0, +\infty)$ ($A \subseteq \mathbb{R}$) is a concave function and

$$0 < \{ \max f(x) \mid x \in A \} \leq M/2,$$

then

$$\left(\frac{\prod_{k=1}^n (f(x_k))^{\lambda_k}}{\prod_{k=1}^n (M - f(x_k))^{\lambda_k}} \right)^{1/\sum_{k=1}^n \lambda_k} \leq \frac{f\left(\frac{\sum_{k=1}^n \lambda_k x_k}{\sum_{k=1}^n \lambda_k}\right)}{M - f\left(\frac{\sum_{k=1}^n \lambda_k x_k}{\sum_{k=1}^n \lambda_k}\right)},$$

where $\lambda_k > 0$, $x_k \in A$ ($k = 1, 2, \dots, n$).

Solution by JOSÉ LUIS DÍAZ-BARRERO, Barcelona, Spain.

Let us consider the function $g(x) = \log\left(\frac{M - f(x)}{f(x)}\right)$. Then

$$g'(x) = -\frac{Mf'(x)}{f(x)(M - f(x))},$$

$$g''(x) = \frac{-Mf''(x)f(x)(M - f(x)) + M(f'(x))^2(M - 2f(x))}{(f(x))^2(M - f(x))^2}.$$

Since $M - f(x) > 0$, $M - 2f(x) > 0$ and $f''(x) < 0$, then $g''(x) > 0$ and $g(x)$ is convex. Applying the weighted version of JENSEN's inequality, namely

$$\sum_{k=1}^n \lambda_k g(x_k) \geq \left(\sum_{k=1}^n \lambda_k\right) g\left(\frac{\sum_{k=1}^n \lambda_k x_k}{\sum_{k=1}^n \lambda_k}\right),$$

where $\lambda_k > 0$, $x_k \in A$ ($1 \leq k \leq n$), we have

$$\sum_{k=1}^n \lambda_k \log\left(\frac{M - f(x_k)}{f(x_k)}\right) \geq \left(\sum_{k=1}^n \lambda_k\right) \log\left(\frac{M - f\left(\frac{\sum_{k=1}^n \lambda_k x_k}{\sum_{k=1}^n \lambda_k}\right)}{f\left(\frac{\sum_{k=1}^n \lambda_k x_k}{\sum_{k=1}^n \lambda_k}\right)}\right),$$

or equivalently

$$\log \left(\prod_{k=1}^n \left(\frac{M - f(x_k)}{f(x_k)} \right)^{\lambda_k} \right)^{1/\sum_{k=1}^n \lambda_k} \geq \log \left(\frac{M - f \left(\frac{\sum_{k=1}^n \lambda_k x_k}{\sum_{k=1}^n \lambda_k} \right)}{f \left(\frac{\sum_{k=1}^n \lambda_k x_k}{\sum_{k=1}^n \lambda_k} \right)} \right).$$

Since the logarithmic function is injective, then

$$\left(\prod_{k=1}^n \left(\frac{M - f(x_k)}{f(x_k)} \right)^{\lambda_k} \right)^{1/\sum_{k=1}^n \lambda_k} \geq \frac{M - f \left(\frac{\sum_{k=1}^n \lambda_k x_k}{\sum_{k=1}^n \lambda_k} \right)}{f \left(\frac{\sum_{k=1}^n \lambda_k x_k}{\sum_{k=1}^n \lambda_k} \right)}.$$

Inverting terms in the preceding inequality the statement immediately follows and we are done.

Also solved by LI ZHOU (whinter Haven, USA) and the proposers.

• **Problem 61** proposed by MIHÁLY BENCZE and MARIAN DINCA, Brasov, Romania. (A generalization of KY FAN's inequality)

If $A_1 A_2 \dots A_n$ is a convex polygon, prove that

$$\sum_{k=1}^n a_k^2 \geq \sum_{k=1}^n a_k a_{k+1} \cos A_k + 4 \text{Area} [A_1 A_2 \dots A_n] \operatorname{tg} \frac{\pi}{n},$$

where a_1, \dots, a_n are lengths of sides, with $a_{n+1} = a_1$.

Solution by proposers.

First we prove the following: *If $A_1 A_2 \dots A_n$ is a convex polygon, B_k denotes the midpoint of sides $A_k A_{k+1}$ ($k = 1, 2, \dots, n$) ($A_{n+1} = A_1$), then*

$$\text{Area} [B_1 B_2 \dots B_n] > \frac{1}{2} \text{Area} [A_1 A_2 \dots A_n].$$

Proof. Since $A_k(a_k)$, $B_k\left(\frac{a_k + a_{k+1}}{2}\right)$ ($k = 1, 2, \dots, n$), we have

$$\begin{aligned} \text{Area}[B_1 B_2 \dots B_n] &= \frac{1}{2} \text{Im} \sum_{k=1}^n \overline{b_k} b_{k+1} = \frac{1}{2} \text{Im} \sum_{k=1}^n \left(\frac{\overline{a_k} + \overline{a_{k+1}}}{2} \right) \left(\frac{a_{k+1} + a_{k+2}}{2} \right) \\ &= \frac{1}{2} \text{Im} \sum_{k=1}^n \frac{\overline{a_k} a_{k+1}}{4} + \frac{1}{2} \text{Im} \sum_{k=1}^n \frac{\overline{a_{k+1}} a_{k+2}}{4} \\ &\quad + \frac{1}{2} \text{Im} \sum_{k=1}^n \frac{\overline{a_k} a_{k+2}}{4} + \frac{1}{2} \text{Im} \sum_{k=1}^n \frac{\overline{a_{k+1}} a_{k+1}}{4} \\ &= \frac{1}{4} \text{Area}[A_1 A_2 \dots A_n] + \frac{1}{4} \text{Area}[A_1 A_2 \dots A_n] \\ &\quad + \frac{1}{4} \text{Area}[A_1 A_2 \dots A_n] + \frac{1}{4} \text{Area}[A_1 A_2 \dots A_n] \\ &> \frac{1}{2} \text{Area}[A_1 A_2 \dots A_n], \end{aligned}$$

so

$$\sum_{k=1}^n B_k B_{k+1}^2 \geq 4 \text{Area}[B_1 B_2 \dots B_n] \text{tg} \frac{\pi}{n}$$

or

$$\begin{aligned} \sum_{k=1}^n \left(\frac{a_k^2}{4} + \frac{a_{k+1}^2}{4} - \frac{2a_k a_{k+1}}{4} \cos A_k \right) &\geq 4 \text{Area}[B_1 B_2 \dots B_n] \text{tg} \frac{\pi}{n} \\ &\geq 2 \text{Area}[A_1 A_2 \dots A_n] \text{tg} \frac{\pi}{n}, \\ \sum_{k=1}^n a_k^2 &\geq \sum_{k=1}^n a_k a_{k+1} \cos A_k + 4 \text{Area}[A_1 A_2 \dots A_n] \text{tg} \frac{\pi}{n}. \end{aligned}$$